# Oscillations of a String Lab 11

Equipment SWS, mechanical wave driver, meter stick, scale, elastic cord, pulley on rod, 2 rods 30 cm or more, 2 large table clamps, set of weights, weight hanger, leads [180 cm (2)], stroboscope

## 1 Introduction

A taught string can be made to vibrate. If the string is plucked and then left alone the oscillations of the string are "free oscillations." If the string is subjected to a time dependent external driving force the oscillations are "forced." The coupling between the string and the driving force can vary continuously from weak to strong. In this experiment the free and forced oscillations of a string will be studied. The free oscillations of a string fixed at both ends will be investigated by driving the string very weakly with a force that varies sinusoidally with time. It will be assumed that the coupling is so weak that the free vibrations are unaffected by the driving force. The forced vibrations of a string fixed at one end and driven at the other end will also be studied. The displacement of the non-fixed end will be sinusoidally varied in the direction transverse to the string.

## 2 Remarks

If a system is displaced from its equilibrium configuration and let go, it will vibrate. The system will vibrate in one or more of its normal modes, or eigenmodes, or modes for short. In a given mode all parts of the system vibrate sinusoidally at the same frequency and pass through their equilibrium positions at the same time, i.e., have the same phase. Except for degeneracy, each mode has a distinct frequency or eigenfrequency. There are as many normal modes as there are degrees of freedom of the system. A continuous system has an infinite number of modes. Any oscillations of a system can be described by a linear combination of its normal modes. If a system is weakly coupled to a sinusoidal external driving force it is found that the system response is much larger at the normal mode frequencies than at other frequencies, assuming that the driving force is applied at an appropriate point.

If a system has boundaries the normal modes are affected by the boundary conditions. One example is a string. If a string is fixed at both ends it has one set of modes. If one end is fixed and the other end is free to slide on a frictionless transverse rod, the string will have a different set of normal modes. Another example is the air inside a pipe, e.g., a wind instrument. The modes are different if an end of the pipe is closed or open.

If a system is driven or forced at a particular frequency, the system oscillates at that frequency (Recall that for free oscillations only the normal mode frequencies can appear.) The response of the system does depend on the normal mode structure of the system and also on exactly how and where the driving force is applied.

## 3 Theory: Free Oscillations

We consider a string fixed at both ends and assumed to vibrate in the $x-y$ plane. See Fig. 1. The length along the string is designated by $x$, with the string fixed at $x = 0$ and $x = L$.

The transverse displacement of the string is called $y = y(x,t)$. The string is assumed to have a uniform constant tension $T$ and a uniform mass per unit length of $\rho$. If Newton's 2nd law in the $y$ direction is applied to a length $dx$ of the string, and if no damping is assumed, the equation describing the string is found to be the wave equation,

$$\frac{\partial^2 y}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = 0. \tag{1}$$

The most general solution to this equation is $g(x \pm vt)$, where $g$ is any function and $v = \sqrt{T/\rho}$. These solutions are waves traveling in the positive or negative $x$ direction with speed $v$. The boundary conditions of our problem limit the possible solutions. We look for sinusoidal normal mode solutions and guess a solution of the form $y = f(x)\cos\omega t$, where $\omega$ is the angular frequency and $t$ is the time. We shortly find that only certain discrete values of $\omega$ satisfy the boundary conditions. Substituting this assumed solution into Eq.(1),

$$\frac{d^2 f}{dx^2} + \frac{\rho}{T}\omega^2 f = 0. \tag{2}$$

The solution to this equation is a sum of sines and cosines, but only the sines will satisfy the boundary condition $y(0,t) = 0$. Assume $f(x)$ to be of the form $A\sin\frac{2\pi}{\lambda}x$, where $A$ and $\lambda$ are constants. The boundary condition $y(L,t) = 0$ can only be satisfied if

$$\lambda_n = \frac{2L}{n}, \qquad n = 1, 2, 3, \ldots, \tag{3}$$

where the subscript on $\lambda$ indicates which of the allowed values it has. A given $n$ refers to a normal mode, and we also subscript $A$ as $A_n$. If $A_n \sin\frac{2\pi}{\lambda_n}x$ is substituted into Eq.(2) we can find the allowed or normal mode angular frequencies $\omega_n$. Using frequency $\nu$ in Hz rather than $\omega$ in radians/s, from $\omega = 2\pi\nu$ the normal mode frequencies are

$$\nu_n = \frac{n}{2L}\sqrt{\frac{T}{\rho}}. \tag{4}$$

The normal mode vibrations of a string are often called standing waves. Standing waves are produced on a string if two sinusoidal waves of the same frequency and amplitude are traveling on the string in opposite directions. There are stationary points on the string separated by $\lambda/2$ where the string does not vibrate, or $y = 0$. These points are called nodes. The 2 fixed end points of a string are nodes. For the lowest frequency or 1st mode, there are no other nodes. Mode 2 has one extra node, mode 3 has 2 extra nodes, and so forth. The first 4 modes are shown in Fig. 2. The solid line shows the string at an instant in time when the amplitude is maximum and all points of the string are instantaneously at rest. A quarter of a period later the string is flat but all points of the string except the nodes are moving. A half period later the string is shown by the dotted line. The envelope of the motion, what you actually see in the lab due to the finite time resolution of your senses, consists of both the dotted and solid lines. The antinodes are those points along the string which have maximum amplitude. The antinodes are situated midway between the nodes.

For a given sinusoidal wave on the string, the usual wave relationship $\lambda\nu = v$ applies. If $g(x \pm vt)$ is substituted into Eq.(2), it is found that the velocity of waves on the string is $v = \sqrt{T/\rho}$. For a given $T$ and $\rho$, the speeds of the waves are a constant independent of the frequency or wavelength. The waves are said to be dispersionless.

In the experiment, the normal modes will be excited by pushing the side of a vibrating rod lightly against the string. The rod will be applied near one end of the string, and the coupling is due to friction between the rod and string. Due to damping of the string, which has not been taken into account in the theory, there will be a response of the string for frequencies slightly different from the normal mode frequencies, but the response will fall off in a typical resonance curve fashion as the applied frequencies moves further from the resonant frequencies.

# 4 Experiment: "Free" Oscillations

The free is in quotes as the oscillations are in fact driven, but weakly enough so that the normal modes will not be substantially affected. Fig. 3 is a sketch of the apparatus. The string used is actually an elastic cord which is quite visible and produces good patterns. One end of the cord is wound several times around a fixed horizontal rod and secured with a few half hitches. The other end of the cord goes over a pulley and has a mass hanger with masses attached to the end. A vibrator, driven by SWS equipment, sits on the bench. The rod of the vibrator is vertical and oscillates up and down. The height of the cord should be adjusted so that the cord is about half-way between the top of the vibrator body and the shoulder of the rod. Adjust the position of the vibrator so that the center of the vibrator rod is 5 cm from the center of the fixed rod and so that the vibrator rod just touches the cord. Now move the vibrator so that its rod moves the cord 0.5 cm in the transverse direction.

The vibrator is driven by the SWS power amplifier, which is driven by the SWS signal generator. You will be using sine waves with frequencies between 5 and 100 Hz. The output voltage of the amplifier should be about 3 or 4 V for these experiments. Be sure the lock on the vibrator is off when you supply voltage.

In determining the frequency of a resonance, a judgment has to be made as to how fine a frequency interval is worth while measuring. Consider 0.5 Hz as a starting point. Too fine an interval requires too many points and takes too long. Too coarse an interval does not give an accurate value of the resonant frequency.

## 4.1 Preliminary Measurements

These are the measurements necessary in order to calculate the normal mode frequencies from Eq.(4). Remove the cord from the apparatus, leaving the small loop in one end for the weight hanger. Weigh the cord and measure its unstretched length. Calculate the unstretched mass per unit length $\rho_0$ of the cord.

Measure $L$, the distance between the center of the fixed rod and the top of the pulley. Put the cord back on the apparatus and, with no tension on the cord, measure the length of the cord from the top of the pulley to the small loop in the cord. Hang a mass of $M$ =400 g from the cord (this includes the mass of the weight hanger) and measure how much the cord stretches. From these measurements you can calculate the mass density $\rho$ of the cord when it has been stretched by any mass $M$, assuming the the cord obeys Hook's law.

## 4.2 Dependence of $\nu_1$ on $T$

Put a mass of 100 g on the cord (reminder: measure any stretch of the cord). Supply a 4 V sine wave to the vibrator and vary the frequency until mode 1 is at a maximum. Record this frequency. Repeat for masses of 200 g and 400 g on the cord. Calculate the frequencies predicted from Eq.(4). Make a table of your experimental and theoretical results. How well to they agree?

### 4.3 Dependence of $\nu_n$ on Mode

Put 200 g on the end of the cord. Find the experimental normal mode frequencies for modes 1-4. Are these frequencies related in the way you expect from Eq.(4)?

### 4.4 Between Modes

With 200 g on the end of the cord, investigate the response of the system for frequencies between $\nu_1$ and $\nu_2$. Record the maximum amplitude as a function of frequency.

## 5 Turn On

Set the string vibrating in the 1st mode. Turn the vibrator off and wait until the string stops oscillating. Then turn the vibrator on and note that it takes some time for the oscillations to build up. About how long does it take? Do you think this time is related to the damping in the system?

## 6 Theory: Forced Oscillations

Consider a taught string fixed at the end $x = 0$ so that $y(0, t) = 0$. The other end at $x = L$ is attached to a mechanical oscillator that moves this end sinusoidally so that $y(L, t) = B \cos \omega t$. Our starting point is Eq.(1). Again assume a solution $y = f(x) \cos \omega t$ which satisfies the boundary condition at $x = 0$. Unlike the free oscillations just discussed, the value of $\omega$ in this assumed solution is known, chosen by the experimenter. Substituting this solution into Eq.(1), Eq.(2) is obtained. Again assume that $f(x)$ is of the form $A \sin \frac{2\pi}{\lambda} x$. If this solution is inserted into Eq.(2) it found found that $\lambda$ must satisfy the wave relation $\lambda\nu = v$. The solution must also satisfy the boundary condition at $x = L$. This condition gives

$$y(L, T) = A \sin \frac{2\pi}{\lambda} L \cos \omega t = B \cos \omega t. \tag{5}$$

B, the driving amplitude, is determined by the experimenter. The unknown is A, and this last equation gives for it

$$A = \frac{B}{\sin \frac{2\pi}{\lambda} L}. \tag{6}$$

The magnitude of A will be a minimum whenever the sine is equal to $\pm 1$. This requires that for a

$$minimum, \qquad \lambda_n^{min} = \frac{4L}{n}, \qquad where \quad n = 1, 3, 5, \dots \; . \tag{7}$$

The frequencies for minima are

$$\nu_n^{min} = \frac{n}{4L} \sqrt{\frac{T}{\rho}}, \qquad where \quad n = 1, 3, 5, \dots \; . \tag{8}$$

A minimum response of the string is obtained by driving at a frequency such that the driving point is an anti-node, or maximum, of the standing wave.

The magnitude of A will be a maximum, actually infinite if there is no damping, whenever the sine is zero. This requires that for a

$$maximum, \qquad \lambda_n^{max} = \frac{2L}{n}, \quad where \quad n = 1, 2, 3, \ldots \quad . \tag{9}$$

The frequencies for maxima are

$$\nu_n^{max} = \frac{n}{2L}\sqrt{\frac{T}{\rho}}, \quad where \quad n = 1, 2, 3, \ldots \quad . \tag{10}$$

According to this criterion, the infinite response of the string is obtained by driving the string at the end of the string when the end is a node. This is a contradiction as the end of the string cannot be a node if it is being driven. This difficulty arises because no damping has been assumed in the theory. For this experiment when there is damping, it is reasonable to assume that maximum response will be obtained for frequencies near those predicted by Eq.(10), and that the best driving point will be near a node close to the end of the string. We omit a theoretical derivation that includes damping.

This theory does predict a continuous change in the string's response as the frequency is changed monotonically. The response goes through a series of minima and maxima, the amplitude of a minimum response being just the driving amplitude.

### 6.1 Unexpected Mode

The string will be driven in the vertical direction and would be expected to vibrate in a vertical plane. In the following experimental section you may find that at and near resonance the string has horizontal as well as vertical oscillations. Look for this by looking at the string not only from the side but from the top. When the string vibrates this way each element of the string is moving in a circle about the equilibrium position of the string. Use the stroboscope to examine the motion. You should take the maximum amplitude of this motion as an indication of resonance. The frequency is close to or the same as the frequency of the mode in which the string vibrate in a vertical plane.

A possible explanation for this behavior is the following. The damping of the string is due not only to air resistance but also the stretching and contraction of the cord. In the circular modes the string does not expand and contract, therefore has less damping, and is more easily excited. Recall also that an oscillating linear displacement can be produced by two counter rotating circular displacements.

## 7 Experiment: Forced Oscillations

See Fig. 4. Put 200 g on the end of the cord. Keep the vibrator rod 5 cm from the fixed rod. Put the cord in the notch at the top of the vibrating rod and move the vibrator so that when the cord is looked at from above, the cord is straight. Move the fixed rod up or down so that the angle that the cord makes between the fixed and vibrating rods is at about 15 deg with the horizontal. Having made these adjustments, lift the cord out of the vibrator rod's notch and put it back in the notch so as to relieve any longitudinal stress on the rod.

### 7.1 Preliminary Measurement

The length $L$ of the string is now the distance between the vibrating rod and the top of the pulley. Make this measurement.

### 7.2 Resonances

Driving the string with a 1 V sine wave, find the lowest 3 resonances. These are the frequencies at which the response of the string is maximum. Do the resonances look like normal mode vibrations? How close are the resonant frequencies to those predicted by Eq.(10)? How close is the driving point to a node at resonance? Which side of the driving point is the node on (between the vibrating rod and the pulley, or between the vibrating rod and the fixed rod)?

### 7.3 Anti-Resonance

Find the 3 lowest anti-resonances. These are the frequencies at which the response of the string is a minimum. How do the frequencies compare to those predicted by Eq.(8). Is the driving point (the vibrator rod) at an anti-node?

### 7.4 Between Modes

Investigate the response of the system for frequencies between the 2 lowest resonances. Compare your results to those obtained in section 4.4.

## 8 Finishing Up

Please leave the bench as you found it. Thank you.

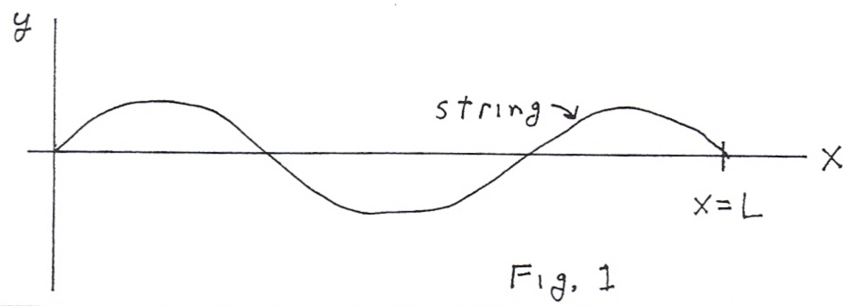


Fig. 1

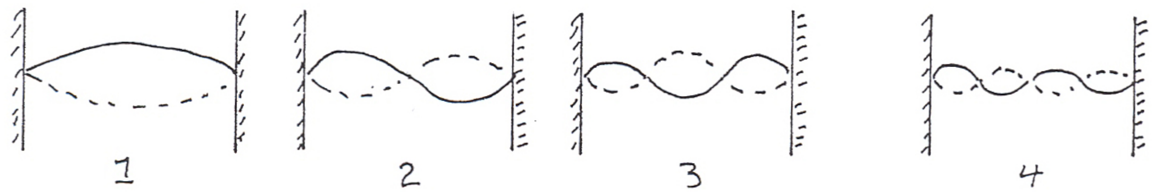


Fig. 2

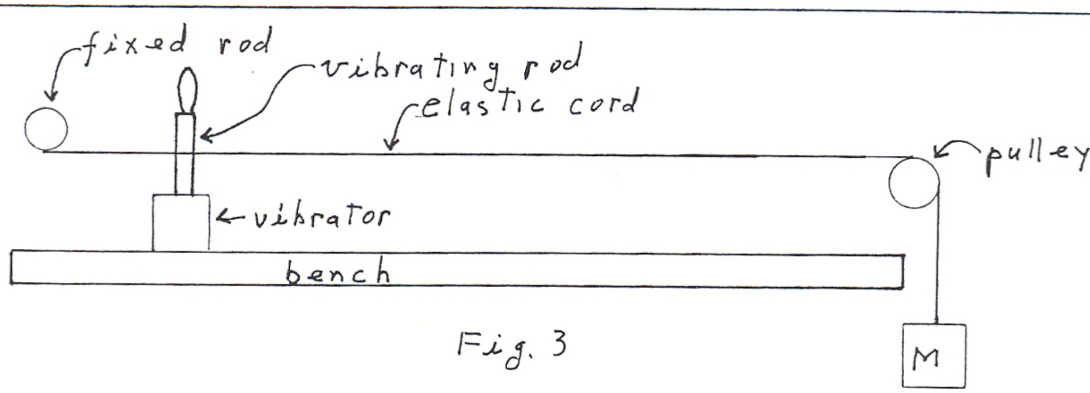


Fig. 3

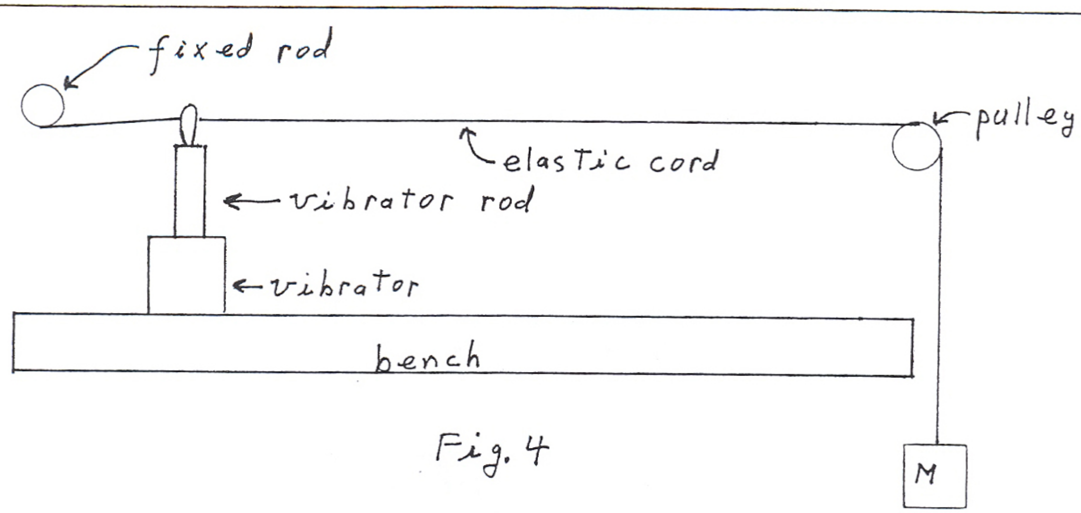


Fig. 4